

On the ϵ -optimality of American options

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Abstract

Purpose – The purpose of this paper is to examine the American pricing problem under the assumption that the holder exercises not only when the underlying asset hits the optimal boundary but also when it falls in a properly defined ϵ -optimal set.

Design/methodology/approach – The purpose of this paper is to examine the American pricing problem under the assumption that the holder exercises not only when the underlying asset hits the optimal boundary but also when it falls in a properly defined ϵ -optimal set. The put and call versions are examined separately. We derive closed-form formulas for the perpetual case. A numerical algorithm for approximating the ϵ -optimal strip is constructed when the maturity is finite. We adapt the Crank–Nicolson finite difference approach to derive the option price or, more precisely, the interval at which it varies. Some numerical experiments are provided.

Findings – We derive closed-form formulas for the perpetual options. A numerical algorithm for approximating the ϵ -optimal strip is constructed when the maturity is finite. We adapt the Crank–Nicolson finite difference approach to derive the option price or, more precisely, the interval in which it varies. Some numerical experiments are provided.

Originality/value – We examine the American pricing problem under the assumption that the holder exercises not only when the underlying asset hits the optimal boundary but also when it falls in a properly defined ϵ -optimal set. This is important to take into account the risk preferences of the investors. Closed-form formulas for the perpetual options are derived. A numerical algorithm for approximating the ϵ -optimal strip is constructed when the maturity is finite. We adapt the Crank–Nicolson finite difference approach to derive the price.

Keywords American options, ϵ -optimality, Perpetual options, Critical values, Crank–Nicolson finite difference method

Paper type Research paper

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1. Introduction

The American options are one of the most important financial instruments against financial risk. This is at most true in the last 2 decades when many catastrophic scenarios happened such as the global financial crisis of 2008, several devastating earthquakes (Japan, Turkey), the COVID-19 pandemic, the wars in Ukraine and the Middle East, etc. This causes the investors to look for non-usual financial instruments to hedge the arising risks. As a result, the scientists turn to these novel derivatives as well as suggest other instruments which to be listed in the markets in the future. On the other hand, the American style derivatives provide several advantages over the relative European ones due to their early exercise right. Some fundamental results for the American options can be found in the seminal works of Jacka (1991) and Kim (1990). Many new investigations appeared recently. Power options under the non-dividend assumptions are studied in Lee (2020). The optimality of the early exercise near maturity is discussed in Battauz *et al.* (2022a). A model under the stochastic interest rates is presented in Battauz and Rotondi (2022). Some stochastic volatility models are considered in Deng (2020), Jeon *et al.* (2020), and Yan *et al.* (2022). The pricing task for the American options under

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regime-switching is studied in [Song et al. \(2022\)](#), [Shirzadi et al. \(2023\)](#), and [Huang et al. \(2024\)](#). The American quanto options are examined in [Battauz et al. \(2022b\)](#). The better-of options are studied in [Jeon and Kim \(2022a\)](#). Options with asset-dependent discounting are considered in [Al-Hadad and Palmowski \(2024\)](#). The so-called variable annuity contracts are studied via the arising variational inequalities in [Jeon and Kim \(2023\)](#). Some first-touch digitals are considered in [Lee et al. \(2024\)](#). The cancelable American options are discussed in [Guo et al. \(2020\)](#), [Zaevski \(2022\)](#), [Palmowski and Stepniak \(2023\)](#), and [Zaevski \(2023a\)](#). Other options which lead to two-sided optimal stopping problems are studied in [Jeon and Oh \(2019\)](#), [De Donno et al. \(2020\)](#), [Qiu \(2020\)](#), [Palmowski et al. \(2021\)](#), [Jeon and Kim \(2022b\)](#), and [Zaevski \(2023b\)](#).

We add to this growing literature a new investigation motivated by several practical purposes related to the investor's preferences. In some situations, the investor cannot wait for the underlying asset to reach the optimal boundary, but he/she needs to exercise at some close enough value. Also, some investors with a low-risk aversion may prefer to exercise at a lower level for a call option (or upper for a put one). Generally said, many reasons may lead to a preference for a premature exercise not at the strict optimal value but in the strip around it.

We first need to define in an appropriate mathematical manner this practical task. We shall control the size of the mentioned above strip by some constant ϵ . As smaller is ϵ , as closer to the optimal value has to be the strip and vice versa. It may seem natural to define the strip via the lines that are half of ϵ below and above the optimal boundary. However, this choice exhibits several major shortcomings mainly due to the strong non-linearity of the American option pricing problem. This prompts us to define the ϵ -optimal strip in a manner arising from the real functional analysis. Let us examine the following example for a real function that has a (local) maximum. In this light, if we define the derivative's price as a function over all possible expiration levels, then the American option can be viewed as its maximum and the optimal boundary as the maximizer. Suppose that a smooth function $f(\cdot)$ has a (local) maximum at a point x_0 in some closed interval. It is well known that for every small enough ϵ , there exist constants x_1 and x_2 , such that $x_1 < x_0 < x_2$, $f(x_1) = f(x_2)$, and $x_2 - x_1 = \epsilon$. Namely the interval (x_1, x_2) can be viewed as ϵ -optimal. We define in this paper the ϵ -optimal strip following this functional analogue. We search for two time-functions with a difference ϵ , one above the true optimal boundary and another below it, for which the strategies of exercising at the first hitting to them moments lead to the same financial result. Defined in that way ϵ -optimality makes this property independent of the particular choice of the initial value of the underlying asset – this is a key characteristic that makes the studied task theoretically and practically reasonable.

We state our model under the [Black and Scholes \(1973\)](#) assumptions considering the underlying asset price as a log-normal stochastic process. Also, following [Shiryaev et al. \(1995\)](#), we introduce an additional discount factor that has two meanings. First, this way we can investigate in our framework the losing value derivatives – their main feature is the decreasing during the time payoffs. Second, this discount factor can be viewed as a dividend rate in a model with changed parameters. More precisely, if we denote by r the risk-free rate, by δ the dividend rate, and by λ the additional discount factor, then a (r, λ, δ) -model for the American option pricing problem is equivalent to a $(r - \nu, \lambda + \nu, \delta - \nu)$ -model for an arbitrary value of ν in the sense that both models lead to the same price – we provide a mathematical proof of this statement. This way we can transfer percentage points between the dividend and the loss rates. Another advantage of this approach is the possibility of studying dividend models in a non-dividend framework. Last but not least, this method leads to some computational facilities.

The classical approach for solving the pricing problem for American options is based on a two-dimensional system of integral equations for the optimal boundary and the price – see [Kim \(1990\)](#) or [Jacka \(1991\)](#). This method needs relatively high computational time. Alternatively, we extend the technique of [Zaevski \(2021\)](#) which first approximates the optimal boundary maximizing the holder's financial result. Theoretically, it is based on some hitting properties of the Brownian motion. An outstanding importance of this approach is that it allows finding the

ϵ -optimal boundaries defined in the above sense in addition to the true optimal boundary. As a consequence, this method turns the free boundary problem related to the American options into a differential task stated on a known region (no matter whether the true or ϵ -optimal one).

We study separately the call and put options beginning with the call ones. We investigate mainly the behavior of the lower boundary – we name it ϵ -optimal. The upper boundary of the strip is obtained by ϵ plus the lower one. It turns out that the ϵ -boundary tends to the true optimal value when the time to maturity tends to zero. Also, we derive the perpetual level as the unique solution of a non-linear equation. Note that the true and ϵ -optimal boundaries are time-independent when there are no maturity restrictions due to the Markovian nature of the log-normal process. On the other hand, we suggest a numerical approach for deriving the whole ϵ -boundary. Once we have its approximation, we may obtain the ϵ -range for the option price. Its lower endpoint is the financial result of the ϵ -boundary whereas the upper value is the true American option price. We discuss also the case when ϵ is not a constant, but a time-dependent function. The related prices are obtained through a modification of the Crank-Nicolson finite difference scheme.

The ϵ -optimality for the put options is considered via some symmetrical arguments. Note that now the ϵ -optimal boundary is the upper one. The main arising difference is due to the lower boundedness of the underlying asset by the zero. It leads to the impossibility of the length of the strip to be ϵ for large enough values. We overcome this problem by restricting the lower boundary by the zero. It turns out that for every time to maturity, there exists a critical value for the deviation ϵ , above which the ϵ -optimal boundary is one and the same. We obtain an equation for this critical value.

The plan of the paper is as follows. The base we use later is presented in [Section 2](#). The results for the call and put options are presented in [Sections 3](#) and [4](#), respectively. Some numerical experiments are provided in [Section 5](#). We conclude by [Section 6](#).

2. Preliminaries

2.1 ϵ -optimality for deterministic functions

We define first the ϵ optimality for deterministic functions. Later we shall define it for the optimal stopping problems, arising from the American option pricing tasks, using some analogies.

Definition 2.1. Let a smooth function $f(x)$ be defined on a finite interval and it achieves its maximum at a point x_0 . In addition, we assume that $f(x)$ is strictly increasing before x_0 and decreasing after. We say that the interval $[x_1, x_2]$ is ϵ -optimal, if $x_1 \leq x_0 \leq x_2$, $x_2 - x_1 = \epsilon$, and $f(x_1) = f(x_2)$.

[Figure 2a](#) describes [Definition 2.1](#). The blue curve presents such a function and the red points form the ϵ -optimal interval for $\epsilon = 1$.

Remark 2.1. We have

$$\int_{x_1}^{x_2} f'(u) du = 0 \quad (2.1)$$

under the assumptions of [Definition 2.1](#).

We have the following characterization:

Corollary 2.1. The singleton $\{x_0\}$ can be viewed as 0-optimal.

We prove below a result related to the uniqueness of the ϵ -optimality.

Proposition 2.1. The ϵ -optimal interval is unique, if it exists, under the assumptions of Definition 2.1.

Proof

Suppose that there exist two pairs $\{x_1^*, x_2^*\}$ and $\{\bar{x}_1, \bar{x}_2\}$ which satisfy the conditions of Definition 2.1. We can think without loss of generality that $x_1^* < \bar{x}_1$. Hence, we have the order $x_1^* < \bar{x}_1 \leq x_0 \leq x_2^* < \bar{x}_2$ since $x_2^* - x_1^* = \epsilon$ and $\bar{x}_2 - \bar{x}_1 = \epsilon$. Using the behavior of the function $f(x)$ we see

$$f(x_1^*) < f(\bar{x}_1) = f(\bar{x}_2) < f(x_2^*) = f(x_1^*). \quad (2.2)$$

The contradiction finishes the proof. $\square$

2.2 Payoffs

We shall discuss now the relation between the dividends and the above-mentioned additional discount factor that can be viewed also as a derivative's loss rate. Let the asset price process be denoted by S_t and the main payoff be defined by the function $G(x)$. Thus if the discount rate is λ , then the derivative pays amount of $N(\zeta, S_\zeta) = e^{-\lambda\zeta} G(S_\zeta)$ if it expires at the possibly random moment ζ . If the derivative is American-styled, then ζ is chosen by the holder. If we have an option with a strike price K , then the total payoff is given by

$$\begin{aligned} N(t, x) &\equiv e^{-\lambda t} G(x) = e^{-\lambda t} (x - K)^+ - \text{call} \\ N(t, x) &\equiv e^{-\lambda t} G(x) = e^{-\lambda t} (K - x)^+ - \text{put.} \end{aligned} \quad (2.3)$$

The function $G(\cdot)$ is $(x - K)^+$ for the call options and $(K - x)^+$ for the puts. The following lemma gives their C^2 -approximations.

Lemma 2.1. The functions $G_c(x; \mu)$ and $G_p(x; \mu)$ defined as

$$\begin{aligned} G_c(x; \mu) &= \begin{cases} x - K - \mu, & \text{for } x \geq K + \mu \\ \frac{(x - K)^2}{2\mu} - \frac{\mu}{2}, & \text{for } x \in (K, K + \mu) \\ -\frac{\mu}{2}, & \text{for } x \leq K \end{cases} \\ G_p(x; \mu) &= \begin{cases} -\frac{\mu}{2}, & \text{for } x \geq K \\ \frac{(x - K)^2}{2\mu} - \frac{\mu}{2}, & \text{for } x \in (K - \mu, K) \\ K - x - \mu, & \text{for } x \leq K - \mu \end{cases} \end{aligned} \quad (2.4)$$

are C^2 -approximations of payoffs (2.3) in the sense that $G_c(x; \mu) \rightarrow (x - K)^+$ and $G_p(x; \mu) \rightarrow (K - x)^+$ when $\mu \rightarrow 0$ and they are twice differentiable.

The following proposition shows how a value can be transferred between the discount and dividend rates:

Proposition 2.2. Let us mean by a (r, λ, δ) -model a mathematical formulation of a financial market that consists of an asset and a derivative (no matter European- or American-styled) written on it under the hypotheses that the risk-free rate is r , the dividend rate is δ , and the additional discount factor is λ . Let ν be a constant such that $-\lambda \leq \nu \leq \delta$. Given these assumptions, the (r, λ, δ) -model is equivalent to a $(r - \nu, \lambda + \nu, \delta - \nu)$ -model in the sense that both of them lead to the same price for the derivative at the initial moment. More precisely, suppose that the derivative expires at some (possibly random) moment ζ and $V(t, x; r, \lambda, \delta)$ is its price at the moment t if the spot value is $S_t = x$. Then the following relation holds

$$V(t, x; r, \lambda, \delta) = e^{\nu t} V(t, x; r - \nu, \lambda + \nu, \delta - \nu). \quad (2.5)$$

Proof

Let us consider the (r, λ, δ) -model. We know that the process $e^{-(r-\delta)t} S_t$ is a martingale w.r.t. the risk-neutral measure. Having in mind that the payoff at the moment ζ is of the form $e^{-\lambda \zeta} G(S_\zeta)$, we find for the derivative's price

$$\begin{aligned} V(t, x; r, \lambda, \delta) &= \mathbb{E}^{t,x} [e^{-r(\zeta-t)} (e^{-\lambda \zeta} G(S_\zeta))] \\ &= e^{\nu t} \mathbb{E}^x [e^{-(r-\nu)(\zeta-t)} (e^{-(\lambda+\nu)\zeta} G(S_\zeta))] \\ &= e^{\nu t} V(t, x; r - \nu, \lambda + \nu, \delta - \nu). \end{aligned} \quad (2.6)$$

The expectation above is namely the price of the derivative under the $(r - \nu, \lambda + \nu, \delta - \nu)$ -model because the difference between the risk-free and dividend rates is not changed (it is again $r - \delta$) and thus the risk-neutral condition still holds. We finish the proof having in mind that the set of the possible strategies for the holder of an American-style derivative is preserved. $\square$

2.3 The model

Let the underlying asset be described by the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t \quad (2.7)$$

with initial condition S_0 under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$. B_t defines a Brownian motion in this space. We assume that the measure $\mathbb{Q}$ is risk neutral, i.e. all price processes are martingales after discounting w.r.t. the money account. We assume that the risk-free rate is the constant r during the option life – note that negative values are possible. Let the additional discounting be at rate λ – we impose $\lambda \geq 0$ as well as $r + \lambda > 0$. It is well-known that early exercising is never optimal for a call option when the dividend rate is zero – in our assumptions when $\lambda = 0$, see for example [Proposition 3.1](#) from [Zaevski \(2021\)](#). For this reason, we assume $\lambda > 0$ when we consider call options; we shall discuss also what differences arise when $\lambda = 0$.

Let us denote the infinitesimal generator of the process (2.7) by $\mathcal{A}$. It is the following second-order differential operator:

$$(\mathcal{A}f)(x) = rx f'(x) + \frac{\sigma^2 x^2}{2} f''(x). \quad (2.8)$$

We define also another differential operator over the C^2 functions as

$$(\mathcal{B}g)(x) = (\mathcal{A}g)(x) - (r + \lambda)g(x). \quad (2.9)$$

Let us denote by $\mathcal{T}_{[t,T]}$ the set of all stopping times ζ with values in the interval $[t, T]$. Note that the mathematical and stochastic concept of the stopping times is associated with the possible holder's strategies for the exercising moment. If the spot price at a moment t is $S_t = x$ and option's holder chooses a strategy ζ , then its financial result is given by the function

$$M(t, x; \zeta) = \mathbb{E}^{t,x} [e^{-r(\zeta-t)} N(\zeta, S_\zeta)]. \quad (2.10)$$

We shall use also the notation $M(t, x; f(\cdot))$ if the stopping time is the first hitting moment to the function $f(t)$ – we shall denote sometimes this stopping time by $\zeta^{f(\cdot)}$. Analogously, we shall use the notation $\zeta^{g(\cdot), f(\cdot)}$ for the first exit from the strip determined by the functions $g(\cdot) \leq f(\cdot)$.

The value function of the optimal stopping problem is

$$V(t, x) = \sup_{\zeta \in \mathcal{T}_{[t,T]}} M(t, x; \zeta). \quad (2.11)$$

Thus the optimal set Y , formed by the points (t, x) at which the immediate exercise is the best holder's strategy, can be defined as

$$Y = \{(t, x) : V(t, x) = N(t, x)\}. \quad (2.12)$$

In some cases, the parametrization w.r.t. the time to maturity $\tau = T - t$ instead of the original (t, T) –one is more convenient. This is possible due to the Markovian feature of the log-normal process. We shall write $(x; \tau)$ instead (t, x) to distinguish the notations. Let us denote by $c(\tau)$ the optimal boundary. The next two propositions describe it as well as the optimal region.

Proposition 2.3. Let the constants p and q be defined as

$$\begin{aligned} p &= 2\sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} \\ q &= \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right). \end{aligned} \quad (2.13)$$

Note that $p \geq q + 1$ and the equality holds only when $\lambda = 0$.

- (1) Let us consider call options. If $\lambda = 0$, then $c(\cdot) = \infty$. If $\lambda > 0$, then $c(\tau)$ is an increasing function with endpoints

$$c(0) \equiv K^* = \max\left(\frac{r+\lambda}{\lambda}, 1\right)K \quad (2.14)$$

$$c(\infty) = \frac{p-q}{p-q-1}K. \quad (2.15)$$

Note that $p > q + 1$ when $\lambda > 0$

(2) The optimal boundary for a put option is a decreasing function with end points

$$c(0) \equiv K^* = \min\left(\frac{r + \lambda}{\lambda}, 1\right)K \quad (2.16)$$

$$c(\infty) = \frac{q}{q + 1}K. \quad (2.17)$$

Proof

For proof see [Propositions 3.4](#) and [3.5](#) from [Zaevski \(2021\)](#) as well as theorems 6.1 and 6.2 from the same reference. $\square$

Proposition 2.4. The optimal region of an American call option is $\Upsilon = \{(x; \tau) : x \geq c(\tau)\}$. On the contrary, the points below the optimal boundary form the optimal region of a put option – $\Upsilon = \{(x; \tau) : 0 < x \leq c(\tau)\}$.

Proof.

See [Propositions 3.2](#) and [3.3](#) from [Zaevski \(2021\)](#). $\square$

2.4 Pricing as a differential task

The early exercise right, the basic feature of the American options, leads to an optimal stopping problem – the investor has to identify the proper moment to use the option. It is well-known that these tasks lead to the so-called free boundary differential problems – we know the equation but we need to find its solution as well as the region where it holds – for more details see for example [van Moerbeke \(1974\)](#) or [Peskir and Shiryaev \(2006\)](#). The American options are studied in this framework in the seminal works of [Kim \(1990\)](#) and [Jacka \(1991\)](#). The pricing problem is solved by a two-dimensional dynamical system of integral equations for the optimal boundary and the price – this system has to be solved numerically. We use an alternative method based on some first-hitting properties of the Brownian motion. Using them, we approximate the optimal boundary maximizing the holder's financial result. Thus the free boundary differential task turns into a boundary value problem in a known region. Our method for call options is summarized in the following algorithm:

- (1) We device the time interval into the grid $0 \equiv t_0 < t_1 < \dots < t_n \equiv T$ and let $D_0, D_1, \dots, D_n$ be some values for the underlying asset at these nodes. We shall approximate the optimal boundary maximizing the holder's financial result at this grid deriving the appropriate values of D 's.
- (2) We use the closed-form formula for the price of a European style option that expires when the underlying asset reaches an exponent of the piecewise linear function that relates the values $D_0, D_1, \dots, D_n$ at the grid – see [Formula \(3.21\)](#). We denote this price by $V(x; \{t_0, \dots, t_n\}; \{D_0, \dots, D_n\})$.
- (3) The optimal boundary at the maturity D_n is given by [Formula \(2.14\)](#).
- (4) Suppose that we know the values of the optimal boundary at the moments $t_m, t_{m+1}, \dots, t_n$ – we denote them by $D_m, D_{m+1}, \dots, D_n$ – for some $m < n$. For a fixed value x of the asset price at the previous moment t_{m-1} , we define $d(x)$ as the value for the boundary that maximizes the holder's financial result:

$$d(x) = \operatorname{argmax}\{D : V(x; \{0, t_m - t_{m-1}, \dots, t_n - t_{m-1}\}; \{D, D_m, \dots, D_n\})\}. \quad (2.18)$$

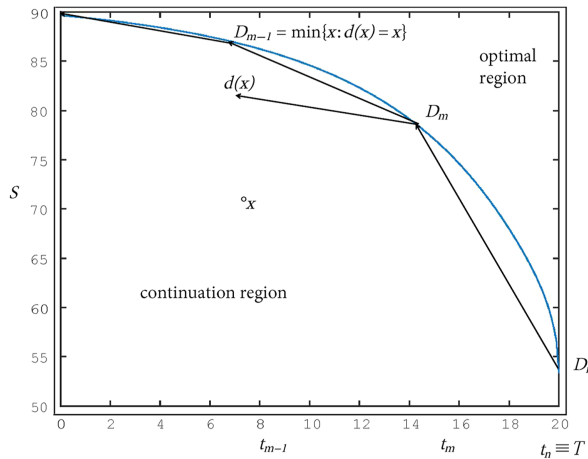
- (5) We approximate the value of the optimal boundary at the previous time-node t_{m-1} as $D_{m-1} = \min\{x : D(x) = x\}$. In fact, this is the lower value for which the immediate exercise is optimal.

The algorithm is illustrated in [Figure 1](#). For more details, see [Section 5](#) of [Zaevski \(2021\)](#). In addition, we shall see later that this approach allows approximating the desired ϵ -strip.

Let us discuss the pricing problem for a derivative which expires if the underlying asset leaves a strip $(A(t), B(t))$, $A(t) < B(t)$. The derivative price – let us denote it by $F(t, x)$ – has to solve the following boundary value problem

$$\begin{aligned} F_t(t, x) + rxF_x(t, x) + \frac{1}{2}\sigma^2 x^2 F_{xx}(t, x) - rF(t, x) &= 0 \\ F(t, A(t)) &= e^{-\lambda t} G(x), \quad t \in (0, T) \\ F(t, B(t)) &= e^{-\lambda t} G(x), \quad t \in (0, T) \\ F(T, x) &= e^{-\lambda T} G(x), \quad x \in (A(T), B(T)). \end{aligned} \tag{2.19}$$

Many numerical methods for solving such kinds of equations are available. We shall use a Crank-Nicolson finite difference approach. It is based on dividing the (t, x) -space into some number of nodes w.r.t. the time as well as w.r.t. the state variable – say M for the time and N for the state variable. The terminal condition is inducted as $F(1, n) = e^{-\lambda T} G(x_n)$ – note that we work backwards. The lower and upper boundary conditions are imposed by setting the price function to $F(m, n) = e^{-\lambda t_m} G(x_n)$ at the points which fall below the lower boundary or above the upper one. For the rest of the points, namely between the boundaries, the Crank-Nicolson scheme is applicable – the derivatives in BVP (2.19) can be approximated using Formulas (B.1); the equation can be written as (B.2); and the values of $F(m, n)$ are the solutions of the appearing after rearranging linear system composed by equations (B.3), (B.4), and (B.5).



Source(s): Author's own work

Figure 1. Approximating the optimal boundary

3. Call style options

Let the function $\overline{G}(x) = (x - K)$ describes the payoff above the strike.

3.1 Definition of ϵ -optimality

We shall say that a function $a(\cdot)$ is less than $b(\cdot)$ if $a(s) \leq b(s)$ for every $s \leq \tau$ and the strong inequality holds at a subset of the interval $(0, \tau)$ with a positive Lebesgue measure. We shall use the notation $a(\cdot) < b(\cdot)$ or $b(\cdot) > a(\cdot)$. If the function $a(\cdot)$ is the constant A , we shall write $A < b(\cdot)$.

We need the following lemma before defining the ϵ -optimality for an American call. It gives the behavior of the option price if we increase the boundary but staying it below the optimal one or decrease it but only above the optimal one.

Lemma 3.1. The inequality $M(x, \tau; a_1(\cdot)) < M(x, \tau; a_2(\cdot))$ holds for every two continuous functions $a_1(\cdot)$ and $a_2(\cdot)$ such that $x < a_1(0)$ and $a_1(\tau) < a_2(\tau) < c(\tau)$. Also, if $c(\tau) < a_1(\tau) < a_2(\tau)$, then $M(x, \tau; a_1(\cdot)) > M(x, \tau; a_2(\cdot))$.

Proof

Suppose the opposite, i.e there exist functions $a_1(\cdot)$ and $a_2(\cdot)$, such that $a_1(\cdot) < a_2(\cdot) < c(\cdot)$, but $M(x, \tau; a_1(\cdot)) > M(x, \tau; a_2(\cdot))$. Let the constant K^* be given by [Formula \(2.14\)](#). We can assume that $K^* < a_1(\cdot)$, because the call payoff is zero below the strike and the operator $\mathcal{B}$ applied to the function $e^{-\lambda t}(x - K)$ changes its sign from positive to negative in the point $\frac{r+\lambda}{\lambda}K$. The assumption above leads to the existence of functions $b(\cdot)$, $b_1(\cdot)$, and $b_2(\cdot)$, such that

$$K^* < a_1(\cdot) < b_1(\cdot) < b(\cdot) < b_2(\cdot) < a_2(\cdot) < c(\cdot), \quad (3.1)$$

and exhibiting the properties

$$M(x, \tau; a(\cdot)) > M(x, \tau; b(\cdot)) \quad (3.2)$$

$$M(x, s; b_1(\cdot)) > M(x, s; b_2(\cdot)) \quad (3.3)$$

for all functions $a(\cdot)$ such that $b_1(\cdot) < a(\cdot) < b(\cdot)$ or $b(\cdot) < a(\cdot) < b_2(\cdot)$. The Dynkin's Formula and inequality (3.2) lead to

$$\begin{aligned} \overline{G}(b(0)) &= \lim_{x \uparrow b(0)} M(x, \tau; b(\cdot)) < \lim_{x \uparrow b(0)} M(x, \tau; b_2(\cdot)) = M(b(0), \tau; b_2(\cdot)) \\ &= \mathbb{E}^{b(0)} \left[e^{-(r+\lambda)\xi^{b_2(\cdot)}} \overline{G}(S_{\xi^{b_2(\cdot)}}) \right] \\ &= \overline{G}(b(0)) + \mathbb{E}^{b(0)} \left[\int_0^{\xi^{b_2(\cdot)}} (\mathcal{B}\overline{G})(S_u) du \right] \end{aligned} \quad (3.4)$$

and thus

$$\mathbb{E}^{b(0)} \left[\int_0^{\xi^{b_2(\cdot)}} (\mathcal{B}\overline{G})(S_u) du \right] > 0. \quad (3.5)$$

Inequality (3.5) leads to

$$\begin{aligned}
0 &< \mathbb{E}^{b(0)} \left[\mathbb{E}^{b(0)} \left[\int_0^{\zeta^{b_2(\cdot)}} (\mathcal{BG})(S_u) du \middle| \mathcal{F}_{\zeta^{b_1(\cdot)}, b_2(\cdot)} \right] \right] \\
&= \mathbb{E}^{b(0)} \left[\int_0^{\zeta^{b_1(\cdot)}, b_2(\cdot)} (\mathcal{BG})(S_u) du \right] \\
&+ \mathbb{E}^{b(0)} \left[I_{\zeta^{b_1(\cdot)} < \zeta^{b_2(\cdot)}} \mathbb{E}^{\zeta^{b_1(\cdot)}, b_1(\zeta^{b_1(\cdot)})} \left[\int_{\zeta^{b_1(\cdot)}}^{\zeta^{b_2(\cdot)}} (\mathcal{BG})(S_u) du \right] \right].
\end{aligned} \tag{3.6}$$

On the other hand, similarly to inequality (3.5), we can prove

$$\mathbb{E}^{\zeta^{b_1(\cdot)}, b_1(\zeta^{b_1(\cdot)})} \left[\int_{\zeta^{b_1(\cdot)}}^{\zeta^{b_2(\cdot)}} (\mathcal{BG})(S_u) du \right] < 0 \tag{3.7}$$

on every sample path for which $\zeta^{b_1(\cdot)} < \zeta^{b_2(\cdot)}$ – we have to use inequality (3.3). Combining inequalities (3.6) and (3.7) we conclude that $(\mathcal{BG})(x)$ has to be positive in a small enough neighborhood of $b(0)$. Hence, $(\mathcal{BG})(b(0)) > 0$. On the other hand $b(0) \geq K^*$ and therefore $(\mathcal{BG})(b(0)) \leq 0$. The contradiction finishes the proof. $\square$

Definition 3.1. We call the function $a(\cdot) < c(\cdot)$ (lower) ϵ -optimal if

$$(a(\tau) - K)^+ = M(a(\tau), \tau; a(\cdot) + \epsilon) \tag{3.8}$$

for every time to maturity τ . The upper boundary is defined as $b(\cdot) = a(\cdot) + \epsilon$.

Combining [Lemma 3.1](#) and [Definition 3.1](#) we obtain the following corollary.

Corollary 3.1. We have $a(\cdot) < c(\cdot) < b(\cdot)$.

We shall prove now that the ϵ -optimal boundary is above the strike.

Proposition 3.1. If $a(\tau)$ is the ϵ -optimal boundary, then $a(\tau) > K$ for $\tau > 0$.

Proof

Suppose the opposite, i.e. $a(\tau) \leq K$ for some τ . Hence $0 = (a(\tau) - K)^+ = M(a(\tau); a(\cdot) + \epsilon)$, which is impossible since $b(\cdot) \geq c(\cdot) > K$ due to [Corollary 3.1](#).

Another corollary that arises is about the asymptotic behavior of the ϵ -optimal boundary when $\epsilon \rightarrow \infty$.

Corollary 3.2. Let the European style call option price with time to maturity τ and initial asset price x be denoted by $p_e(x, \tau)$. Then the limit $\lim_{\epsilon \rightarrow \infty} a(\tau)$ is the unique root of the function $f(y) = p_e(y, \tau) - y + K$.

Proof.

The proof follows the fact that the right hand-side of [Equation \(3.8\)](#) turns to the European option price when $\epsilon \rightarrow \infty$. Note that $a(\tau) > K$ due to [Proposition 3.1](#). The root exists and it is unique because the delta of a call is positive but smaller than one function and the endpoints are $f(K) > 0$ and $f(\infty) < 0$. The last inequality is true since the asymptotic behavior of the function $f(y)$ in the infinity is like $K(1 - e^{-(r+\lambda)\tau}) - y(1 - e^{-\lambda\tau}) < 0$ when $\lambda > 0$. $\square$

Remark 3.1. Let us discuss the case $\lambda = 0$. Obviously, $b(\tau) = \infty$ since $c(\tau) = \infty$ and thus $a(\tau) = \infty$ too. Note that $f(\infty) > 0$ when $\lambda = 0$.

Next we show that the ϵ -optimal boundary does not dependent on the particular initial asset value.

Proposition 3.2. Let the function $a(\tau)$ satisfies the conditions of Definition 3.1 for the lower ϵ -optimal boundary. Then $M(x, \tau; a(\cdot)) = M(x, \tau; b(\cdot))$ for every τ and $x \leq a(\tau)$.

Proof.

Let $a(\cdot)$ and $b(\cdot)$ be the ϵ -optimal boundaries. We have $\zeta^{a(\cdot)} \leq \zeta^{b(\cdot)}$ and if $\zeta^{a(\cdot)} \geq \tau$, then $\zeta^{b(\cdot)} > \tau$. Hence,

$$\begin{aligned}
 M(x, \tau; b(\cdot)) &= \mathbb{E}^x \left[e^{-(r+\lambda)\zeta^{b(\cdot)} \wedge \tau} \left(S_{\zeta^{b(\cdot)} \wedge \tau} - K \right)^+ \right] \\
 &= \mathbb{E}^x \left[\mathbb{E} \left[e^{-(r+\lambda)\zeta^{b(\cdot)} \wedge \tau} \left(S_{\zeta^{b(\cdot)} \wedge \tau} - K \right)^+ \left(I_{\zeta^{a(\cdot)} < \tau} + I_{\zeta^{a(\cdot)} \geq \tau} \right) \middle| \mathcal{F}_{\zeta^{a(\cdot)}} \right] \right] \\
 &= \mathbb{E}^x \left[I_{\zeta^{a(\cdot)} < \tau} \mathbb{E}^{\zeta^{a(\cdot)}, a(\zeta^{a(\cdot)})} \left[e^{-(r+\lambda)\zeta^{b(\cdot)} \wedge \tau} \left(S_{\zeta^{b(\cdot)} \wedge \tau} - K \right)^+ \right] \right. \\
 &\quad \left. + \mathbb{E}^x \left[I_{\zeta^{a(\cdot)} \geq \tau} e^{-(r+\lambda)\tau} (S_\tau - K)^+ \right] \right] \\
 &= \mathbb{E}^x \left[I_{\zeta^{a(\cdot)} < \tau} \left(S_{\zeta^{a(\cdot)}} - K \right) \right] + \mathbb{E}^x \left[I_{\zeta^{a(\cdot)} \geq \tau} e^{-(r+\lambda)\tau} (S_\tau - K)^+ \right] \\
 &= M(x, \tau; a(\cdot)).
 \end{aligned} \tag{3.9}$$

3.2 Deriving the ϵ -optimal boundaries

First we need to derive the endpoints of the ϵ -optimal boundary, namely $a(0)$ and $a(\infty)$. Suppose for a moment that the payoff function $N(t, x)$ has the form $N(t, x) = e^{-\lambda t} G(x)$ for a twice differentiable function $G(x)$. Using the Dynkin's formula, we can see that the lower optimal boundary would be characterized by the equation

$$\mathbb{E}^{t, a(t)} \left[\int_t^{\zeta^{b(\cdot)} \wedge T} (\mathcal{B}G)(S_u) du \right] = 0. \tag{3.10}$$

The following proposition characterizes the value of $a(0)$.

Proposition 3.3. We have $a(0) = c(0) = K^*$, where the constant K^* is given by Formula (2.14).

Proof.

The inequality $a(0) \geq K$ follows [Proposition 3.1](#). Suppose $c(0) > K$ and $K < a(0) < K^*$ – this case is possible only when $r > 0$. Having in mind the twice differentiability of the function $(x - K)^+$ for $x > K$ and taking the limit $t \rightarrow T$ in [Equation \(3.10\)](#), we see $(\mathcal{B}G)(a(0)) = 0$. Therefore $a(0) = c(0) = \frac{r+\lambda}{\lambda} K$, because $(\mathcal{B}G)(x) = -\lambda x + (r + \lambda)K$. If we suppose $a(0) = K$, then we can use the C^2 -approximation given in [Lemma 2.1](#) to see that this is impossible. $\square$

3.3 Perpetual case

In the following proposition, we obtain the value $a(\infty)$ which is the ϵ -optimal boundary for an option without maturity constraints:

Proposition 3.4. The value $a(\infty)$ is the unique solution of the equation

$$\frac{y - K}{y + \epsilon - K} = \left(\frac{y}{y + \epsilon} \right)^{p-q}, \quad (3.11)$$

where the constants p and q are defined by Equations (2.13). It leads to price

$$M(S_0, a(\infty)) = (a(\infty) - K) \left(\frac{S_0}{a(\infty)} \right)^{p-q}. \quad (3.12)$$

Something more, $a(\infty) \rightarrow K$ when $\epsilon \rightarrow \infty$ and

$$a(\infty) \rightarrow c(\infty) = \frac{p-q}{p-q-1} K \text{ for } \epsilon \rightarrow 0. \quad (3.13)$$

Note that this is just the optimal boundary of the American call option.

Proof

As we mentioned before, the ϵ -optimal boundary is flat – the same is true for the real optimal boundary too. Suppose that $a(\infty) = y \geq K$ and the initial asset value is a small enough constant $S_0 = x$. Therefore, the first hitting moment of the asset to the level y can be viewed as the first hitting of the Brownian motion with drift

$$\mu = \frac{r}{\sigma} - \frac{\sigma}{2} \quad (3.14)$$

to the value

$$a = \frac{\ln y - \ln x}{\sigma} > 0. \quad (3.15)$$

Using equation (A.8) we derive for the price, written as a function of y ,

$$\begin{aligned} f(y) &= \mathbb{E}^x \left[e^{-(r+\lambda)\tau} (S_\tau - K) I_{\tau < \infty} \right] \\ &= (y - K) \mathbb{E}^x \left[e^{-(r+\lambda)\tau} I_{\tau < \infty} \right] \\ &= (y - K) \exp \left\{ -(p - q) \frac{\ln y - \ln x}{\sigma} \right\} \\ &= (y - K) \left(\frac{x}{y} \right)^{p-q}. \end{aligned} \quad (3.16)$$

The derivative of function (3.16) is

$$f'(y) = \frac{x^{p-q}}{y^{p-q+1}} [-y(p - q - 1) + K(p - q)]. \quad (3.17)$$

Note that $p > q + 1$ since $\lambda > 0$. Let the constant $\bar{y}$, defined as

$$\bar{y} = \frac{p - q}{p - q - 1} K, \quad (3.18)$$

be the zero of derivative (3.17). We can conclude that the function $f(y)$ starts from the value $f(K) = 0$, increases in the interval $(K, \bar{y})$ having a maximum for $y = \bar{y}$, and decreases for $y > \bar{y}$. Hence, the ϵ -optimal boundary really exists and it is the solution of Equation (3.11) due to the ϵ -optimality condition $f(y) = f(y + \epsilon)$. The uniqueness of the solution follows Proposition 2.1 – note that conditions of Definition 2.1 are satisfied. Also, this boundary leads to a price given by Formula (3.12). Observing that $f(K) = f(\infty) = 0$, we prove that $a(\infty) \rightarrow K$ for $\epsilon \rightarrow \infty$. We finish the proof reminding that the real optimal boundary is reached for $\epsilon \rightarrow 0$ and it is given by the max-point (3.18). $\square$

Remark 3.2. Let us discuss the case $\lambda = 0$. This assumption leads to $p = q + 1$ and hence Equation (3.11) is solved by $y = \infty$. This is in accordance with Remark 3.1.

3.4 Finite maturity horizon

The next step is to approximate the whole ϵ -optimal boundary – we shall use exponents of piecewise linear functions. We devise the time interval $[0, T]$ into n sub-intervals $0 \equiv t_0 < t_1 < \dots < t_n \equiv T$. Suppose that the holder strategy is first hitting to the level $\exp(a_i t + b_i)$ if it happens in the interval $[t_{i-1}, t_i)$, $i = 1, 2, \dots, n$. If this does not happen before the maturity, then the holder receives the payoff at the maturity. We also impose a continuity at the grid nodes, namely $\exp(a_i t_i + b_i) = \exp(a_{i+1} t_i + b_{i+1}) \equiv D_i$. Let the initial value of the underlying asset be $S_0 = x$. Hence, the holder exercises if the Brownian motion touches the level

$$\frac{1}{\sigma} \left(\left(a_i - r + \frac{\sigma^2}{2} \right) t + b_i - \log(x) \right) = A_i t_i + B_i \quad (3.19)$$

for

$$\begin{aligned} A_i &= \frac{1}{\sigma} \left(a_i - r + \frac{\sigma^2}{2} \right) \\ B_i &= \frac{b_i - \log(x)}{\sigma} \end{aligned} \quad (3.20)$$

in the corresponding sub-interval. Let ζ be just this moment. The price of this derivative, $M(x, \tau; \zeta)$, can be derived as

$$\begin{aligned} M(x, \tau; \zeta) &= \mathbb{E}^x \left[e^{-(r+\lambda)\zeta} (S_\zeta - K)^+ \right] \\ &= \mathbb{E}^x \left[e^{-(r+\lambda)\zeta} (S_\zeta - K) I_{\zeta < T} \right] + e^{-(r+\lambda)T} \mathbb{E}^x \left[(S_T - K)^+ I_{\zeta = T} \right] \\ &= x \sum_{m=1}^n \mathbb{E} \left[e^{-\alpha_{2,m}\zeta} I_{t_{m-1} < \zeta \leq t_m} \right] - K \mathbb{E} \left[e^{-\alpha_1 \zeta} I_{\zeta < T} \right] \\ &\quad + x e^{-\alpha_3 T} \mathbb{E} \left[e^{\sigma B_T} I_{B_T > k, \zeta = T} \right] - K e^{-\alpha_1 T} \mathbb{Q}(B_T > k, \zeta = T), \end{aligned} \quad (3.21)$$

where

$$\begin{aligned} \alpha_1 &= r + \lambda \\ \alpha_{2,m} &= (r + \lambda) - \left(r - \frac{\sigma^2}{2} \right) - A_m \sigma = \frac{\sigma^2}{2} - A_m \sigma + \lambda \\ \alpha_3 &= \lambda + \frac{\sigma^2}{2} \\ k &= \frac{1}{\sigma} \log \left(\frac{K}{x} \right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2} \right) T. \end{aligned} \quad (3.22)$$

We approximate the ϵ -optimal boundary iteratively backwards – the algorithm is as follows.

- (1) [Proposition 3.3](#) leads to $D_n = \max(\frac{x+\lambda}{\lambda}, 1)K$.
- (2) Suppose that we know all values $D_m, D_{m+1}, \dots, D_n$.
- (4) Let us denote by ζ^x the first hit of the asset starting at the point x at the moment t_{m-1} to the boundary determined by the points $x + \epsilon, D_m + \epsilon, D_{m+1} + \epsilon, \dots, D_n + \epsilon$. We derive the value of D_{m-1} as the solution of the equation w.r.t. the variable x

$$x - K = M(x, T - t_{m-1}; \zeta^x). \quad (3.23)$$

Remark 3.3. We can easily modify the presented above algorithm if ϵ is time dependent. A natural assumption is that the lower time to maturity, the lower deviation $\epsilon(\tau)$. Something more, the condition $\epsilon(0) = 0$ is quite reasonable. Hence, we can define $\epsilon(\tau)$ as the cumulative distribution function with domain $[0, T]$ multiplied by a constant.

Once we approximate the ϵ -optimal boundary, we can derive an interval for the option price using the Crank-Nicolson scheme presented in [Section 2.4](#). Obviously, its upper boundary is the price of the American option, which can be derived by taking $\epsilon \rightarrow 0$. The upper boundary of BVP (2.19) is namely the ϵ -optimal boundary. The natural choice for the lower one is the zero. The price of the call option at it is zero.

3.5 Algorithm for approximating the ϵ -optimal boundary

We summarize now the algorithm for obtaining the ϵ -optimal boundary and the related strip.

- (1) We define the ϵ -optimal strip by some analogies with the real functions – [Definition 3.1](#).
- (2) We prove a series of statements for the ϵ -optimal boundary – [Lemma 3.1](#), [Corollaries 3.1](#) and [3.2](#), [Propositions 3.1](#) and [3.2](#).
- (3) We obtain the value when time to maturity tends to zero $a(0)$ – [Proposition 3.3](#).
- (4) We derive the perpetual value $a(\infty)$, i.e. when there are no maturity restrictions – [Proposition 3.4](#).
- (5) We approximate the ϵ -optimal boundary at a point T through the following steps:
 - We devise the interval $[0, T]$ into a grid $0 \equiv t_0 < t_1 < \dots < t_n \equiv T$.
 - The initial value of the boundary is derived in point 3 of this algorithm – it is related to the value at the moment $t_n = T$.
 - Suppose that we have derived all values at the points $t_m, t_{m+1}, \dots, t_n = T$ for some $m \leq n$.
 - If the initial asset value is x , then we consider a European style option with time to maturity $T - t_{m-1}$ that expires when the asset reaches the boundary formed by the values $x + \epsilon, a(t_m), a(t_{m+1}), \dots, a(t_n)$ on the grid $0, t_m - t_{m-1}, t_{m+1} - t_{m-1}, \dots, t_n - t_{m-1}$. Remind that this boundary is composed of an exponent of the continuous piecewise linear function that connects these points. We denote the financial result of this strategy by $M(x)$ marking the dependence on the initial asset value x . It can be obtained via [Formula \(3.21\)](#). Our approximation of the ϵ -optimal boundary at the $(m - 1)$ -th node is the solution of the equation $M(x) = x - K$.

- (6) We approximate the whole boundary varying the value of T .
 (7) The upper boundary of the ϵ -optimal strip is obtained as $b(t) = a(t) + \epsilon$.

4. Put style options

We shall follow the algorithm presented in [Section 3.5](#) having in mind some symmetry arguments. The major difference that arises when we deal with put options is related to the lower boundedness of the underlying asset by the zero. We overcome this problem defining the ϵ -optimality via the following modification of [Equation \(3.8\)](#)

$$(K - a(\tau))^+ = M(a(\tau), \tau; \max\{a(\cdot) - \epsilon, 0\}). \quad (4.1)$$

Note that now $c(\tau) < a(\tau)$ and $b(\cdot)$ is defined as $b(\tau) = \max\{a(\tau) - \epsilon, 0\}$. Using some symmetrical arguments we can prove the results for the put options.

Proposition 4.1. The following statements hold.

- (1) The inequality $M(x, \tau; a_1(\cdot)) < M(x, \tau; a_2(\cdot))$ holds for every two continuous functions $a_1(\tau) > a_2(\tau) > c(\tau)$. Also, if $c(\tau) > a_1(\tau) > a_2(\tau)$, then $M(x, \tau; a_1(\cdot)) > M(x, \tau; a_2(\cdot))$.
 (2) We have $a(\cdot) > c(\cdot) > b(\cdot)$.
 (3) The ϵ -optimal boundary $a(\tau)$ is below the strike, $a(\tau) < K$ when $\tau > 0$.
 (4) The equality $M(x, \tau; a(\cdot)) = M(x, \tau; b(\cdot))$ holds for every τ and $x \geq a(\tau)$.
 (5) The initial value of the ϵ -optimal boundary is $a(0) = c(0) = K^*$. The value of K^* is given by Formula (2.16).

We need to introduce an auxiliary function before establishing the behavior of the ϵ -optimal boundary for large values of ϵ . The necessary results are presented in the lemma below.

Lemma 4.1. Let the price of the European style put option be denoted by $p_p(y, \tau)$ – y is the initial asset value, and τ is the time to maturity. Let the function $f(y)$ be defined as

$$f(y; \tau) = p_p(y, \tau) + y - K. \quad (4.2)$$

Function (4.2) has a unique root in the interval $(0, K]$, which we denote by $y(\tau)$. We have that $y(\tau) \rightarrow K^*$ when $\tau \rightarrow 0$ and $y(\tau) \rightarrow K$ when $\tau \rightarrow \infty$.

Proof

The root of function (4.2) is unique, because the delta of a put is negative but larger than minus one, and $f(0) = -K(1 - e^{-(r+\lambda)\tau}) < 0$, whereas $f(K) = p_p(K, \tau) > 0$.

Let us turn to the asymptotics of $y(\tau)$ – first we shall discuss the case $\tau \rightarrow 0$. Suppose that $r \geq 0$ and thus $K^* = K$. The point $(\tau, y(\tau))$ belongs to the continuation region of the put option since $y(\tau)$ is the root of function (4.1). Hence, $y(\tau) \geq c(\tau)$ and thus $y(\tau) \rightarrow K$ when $\tau \rightarrow 0$,

because $c(0) = K$ and $y(\tau) \in (0, K]$. Suppose now that $r < 0$. Using the smooth approximation of the put payoff given in [Lemma 2.1](#), we obtain

$$\begin{aligned}
 0 &= \lim_{\tau \rightarrow 0} \frac{f(y(\tau); \tau)}{\tau} \\
 &= \lim_{\tau \rightarrow 0} \frac{p_p(y(\tau), \tau) + y - K}{\tau} \\
 &= \lim_{\tau \rightarrow 0} \frac{\mathbb{E}^{y(0)}[e^{-(r+\lambda)\tau} G(S_\tau)] - G(y(\tau)) \pm G(y(0))}{\tau} \\
 &= y'(0)[(\mathcal{B}G)(y(0)) + 1].
 \end{aligned} \tag{4.3}$$

Therefore $y'(0) = 0$. This leads to

$$\lim_{\tau \rightarrow 0} \frac{\mathbb{E}^{y(0)}[e^{-(r+\lambda)\tau} G(S_\tau)] - G(y(0))}{\tau} = 0. \tag{4.4}$$

We establish the zero-asymptotics having in mind that the theta of a put is zero only at the point $\frac{r+\lambda}{\lambda}K$.

We finish the proof observing that $p_p(y, \tau) \rightarrow 0$ when $\tau \rightarrow \infty$. $\square$

We can establish now the behavior of the ϵ -optimal boundary $a(\tau)$ for large values of ϵ .

Proposition 4.2. We have that $a(\tau) = y(\tau)$ for every $\epsilon \geq \max_{t \in [0, \tau]} y(t)$. If $r \geq 0$, then this maximum is namely the strike.

Proof.

If $\epsilon \geq y(t)$ for every $t \in [0, \tau]$, then

$$M(y(\tau), \tau; \max\{y(\cdot) - \epsilon, 0\}) = M(y(\tau), \tau; 0) = p_p(y, \tau). \tag{4.5}$$

Having in mind that $y(\tau)$ is the root of function (4.2), we conclude that [Equation \(4.1\)](#) is satisfied for $a(\tau) = y(\tau)$. We finish the proof noting that if $r \geq 0$, then $y(0) = K$, and thus the restriction for ϵ is namely the strike. $\square$

Let us turn to the perpetual case. The ϵ -optimal boundary as well as the option value are obtained below.

Proposition 4.3. The perpetual value of the ϵ -optimal boundary can be derived via the equation

$$\frac{K - y}{K - \max\{y - \epsilon, 0\}} = \left(\frac{\max\{y - \epsilon, 0\}}{y} \right)^q. \tag{4.6}$$

The constant q is defined in equations (2.13). This solution is unique and it leads to price

$$M(S_0, a(\infty)) = (K - a(\infty)) \left(\frac{a(\infty)}{S_0} \right)^q. \tag{4.7}$$

Something more, $a(\infty) \rightarrow K$ when $\epsilon \rightarrow \infty$ and

$$a(\infty) \rightarrow c(\infty) = \frac{q}{q+1}K \text{ for } \epsilon \rightarrow 0. \tag{4.8}$$

This is the optimal boundary of the American put.

Proof

Suppose that the initial asset value is the large enough constant $S_0 = x$ and the holder exercises when the asset reaches the level y . We view again this event as the first hitting moment of the Brownian motion with drift (3.14) to the value (3.15). Note that this value is negative now. Using equation (A.8) we derive the price as a function of the variable y :

$$f(y) = (K - y) \left(\frac{y}{x} \right)^q. \quad (4.9)$$

We deduce Equation (4.6) using the definition of the ϵ -optimality provided in Equation (4.1). This value leads to a put price as in Formula (3.12). The uniqueness of the solution follows Proposition 2.1. The limit $a(\infty) \rightarrow K$ when $\epsilon \rightarrow \infty$ stands, because $f(K) = f(0) = 0$. We finish the proof keeping in mind that the true optimal boundary is reached for $\epsilon \rightarrow 0$. $\square$

We deal with the finite maturity put options using the algorithm established for the calls. There are three main differences – first, the opposite price function (3.21), second, the lower hitting problem, and third, Equation (3.23) now is $K - x = M(x, T - t_{m-1}; \zeta^x)$. We have to discuss another feature of function (3.21) related to the boundedness below of the underlying asset. If for some i and j , such that $i + 2 \leq j \leq n$, $D_i > 0$, $D_j > 0$, and $D_q = 0$ for every $i < q < j$, then the part of Formula (3.21) related to the interval (t_i, t_j) can be derived via the European option prices as

$$\sum_{m=i+1}^j (K \mathbb{E}[e^{-\alpha_1 \zeta} I_{t_{m-1} < \zeta \leq t_m}] - x \mathbb{E}[e^{-\alpha_2, m \zeta} I_{t_{m-1} < \zeta \leq t_m}]) = p_p(x, t_j - t_i). \quad (4.10)$$

Once we approximate the ϵ -optimal boundary, we derive the price as the solution of BVP (2.19). There is a little problem since the continuation region for the put options is open above. We solve it by introducing an auxiliary large enough boundary – it is natural to set the price to zero on it.

5. Numerical results

We provide several numerical examples to illustrate the derived theoretical results. Let us consider first call options. We use the following parameters – risk-free rate $r = 0.01$, additional discount rate $\lambda = 0.05$, volatility $\sigma = 0.3$, and strike price $K = 20$. We vary the deviation term as $\epsilon \in [0.01, 20]$. The endpoints of the true optimal boundary can be calculated via Equations (2.14) and (3.10) – they are $c(0) = \max(\frac{r+\lambda}{\lambda}, 1)K = 24$ and $c(\infty) = \frac{p-q}{p-q-1}K = 52.9317$. The way we derive the lower ϵ -boundary can be seen in Figure A1(a). The blue curve is the price function w.r.t. the boundary value. We search for this a , for which $M(a, a) = M(a, a + \epsilon)$ – in this figure we consider $\epsilon = 1$. The ϵ -optimal values are presented by red circles and they are 34.8392 and 35.8392 – note that the distance between them is namely $\epsilon = 1$.

Both of ϵ -boundaries are presented jointly in Figure A1(b) – the used deviations are $\epsilon \in \{1, 5, 10\}$ and the time to maturity is assumed to be $T = 100$. To see the behavior for some large deviations, we consider also values $\epsilon \in \{20, 50, 100\}$. We present only the lower boundary because the length of the interval is too large – see Figure A1(c). The boundary surface can be viewed in Figure A1(e). We plot by red points the perpetual values and by blue ones the real optimal points. The price behavior is presented in Figure A1(d). The initial asset price is assumed to be $S_0 = \$35$. The meaning of the red and blue points is preserved.

As we mentioned in Remark 3.3, we may assume that the deviation ϵ is a time-dependent function – for example, multiplied by a hundred cumulative distribution function of the μ -intensity exponential distribution

$$\epsilon(\tau) = 100(1 - e^{-\mu\tau}). \quad (5.1)$$

We vary the intensity parameter μ in the interval $\mu \in [0, 10]$. Note that $\epsilon(\cdot) \equiv 0$ when $\mu = 0$ and thus we reach the true optimal boundary. Having in mind that $\epsilon(\infty) = 100$ for every $\mu > 0$, we derive the perpetual ϵ -boundary value using Equation (3.11) – it is 30.8865. The derived results are presented in Figure A1(f). Note that $a(\infty) = 30.8865$ when $\mu > 0$, but $a(\infty) = 52.9317$ when $\mu = 0$ (the perpetual boundary for the American option). We can observe that as small is the intensity μ , as small is the speed of convergence to the presented by the red circles perpetual value. Also, the values for small intensities and maturities are very close to the true optimal levels (the blue circles). On the other hand, the intensity impact is significant for the higher maturities leading to larger deviations between the true and ϵ -boundaries.

Let us consider now several examples for put options. The used parameters are the same – $r = 0.01$, $\lambda = 0.05$, $\sigma = 0.3$, and $K = 20$. We present the ϵ -optimal interval in Figure A2(a) varying the constant ϵ amongst $\{0.5, 1, 2\}$ whereas the results for $\epsilon \in \{10, 15, 20\}$ can be viewed in Figure A2(b). The true optimal boundary is presented by a red curve. The perpetual values are plotted by circles. The surface of the ϵ -optimal boundary can be seen in Figure A2(c) whereas the derived prices are plotted in Figure A2(d). The initial asset price is assumed to be $S_0 = \$19$. The deviation ϵ is varied in the interval $(0, 20]$. The meaning of the red and blue circles is preserved.

Finally, we discuss the critical value for the deviation epsilon, above which the ϵ -optimal boundary is the root of function (4.2) – see also Lemma 4.1. If $r \geq 0$, then this value is the strike. More complicated is the negative case, $r < 0$, equivalent to $K^* < K$. We present the roots of function (4.2) varying the time to maturity in Figure A2(e). The risk-free rate is considered amongst $r \in \{-0.04, -0.02, 0, 0.02, 0.04\}$. The ϵ -optimal boundary at the moment τ is equal to the corresponding root of function (4.2) when $\epsilon \geq y(t)$ for every $t \in [0, \tau]$ – see Proposition 4. The boundary surface when the risk-free rate is $r = -0.01$ can be seen in Figure A2(f). We present by green circles the critical values for ϵ .

6. Conclusions

We have discussed in this paper a sub-optimal exercise of American options motivated by the investor's risk preferences. We defined the so-called ϵ -optimality in the light of some arguments arising from the real functional analysis that produce an appropriate interval around the function's extrema. We translated this concept over the space of investor's strategies and the related financial results. Next, we discussed the call and put options separately. We derived closed-form formulas when the options are not limited by maturity restrictions. Alternatively, we approximated the ϵ -optimal boundaries under the finite maturity horizon and then we applied the Crank-Nicolson finite difference scheme to derive the related sub-optimal prices. The behavior of these boundaries has been obtained. The main difference between the call and put ones arose from the fact that the space of the possible asset prices is limited from below by the zero. Also, we have discussed the ϵ -optimality under the assumption that ϵ is not a constant but a time function. Many numerical results have been provided to illustrate the derived theoretical results.

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Appendices

Appendix 1 First hitting time propositions

Let $c(t) = (a_m t + b_m)I_{t_{m-1} < t \leq t_m}$ be a continuous piecewise linear function w.r.t. the time grid $0 \equiv t_0 < t_1 < \dots < t_n \equiv T$ with node values $c_m = a_m t_m + b_m$. We may assume $c(0) < 0$ due to the symmetry of the Brownian motion. Most of the presented propositions are proven in [Zaevski \(2020\)](#) for the upper hitting

problem, $c(0) > 0$. We shall use the notation $c(t) = at + b$, $b < 0$, if the function is linear. We notate by Λ_t the indicator process $\Lambda_t = I_{t < T}$. Let us denote by $N(\cdot)$ the cumulative distribution function of the standard normal distribution.

Proposition A.1. The probability a Brownian motion to hit a linear function before the moment T is

$$g(T; a, b) \equiv \mathbb{P}(\tau < T) = N\left(\frac{aT + b}{\sqrt{T}}\right) + \exp(-2ab)N\left(\frac{-aT + b}{\sqrt{T}}\right). \quad (\text{A.1})$$

Proof

The proof can be found in Wang and Pötzelberger (1997), Equation (3), Karatzas and Shreve (1991), or in Zaeviski (2020), Proposition 3.1. $\square$

Proposition A.2. The Laplace transform of first hitting to a linear function if it happens before T is

$$L(T, \alpha; a, b) = \mathbb{E}[e^{-\alpha\tau} \Lambda_T] = e^{-b(\sqrt{a^2 + 2\alpha} + a)} g\left(T; -\sqrt{a^2 + 2\alpha}, b\right), \quad (\text{A.2})$$

where $g(\cdot)$ is given by Formula (A.1) and α is an arbitrary positive constant.

Proof

The proof can be found in Zaeviski (2020), theorem 3.1. $\square$

Proposition A.3. If the function is piecewise linear, then the Laplace transform in the interval (t_{m-1}, t_m) is

$$\begin{aligned} & \mathbb{E}\left[e^{-\alpha\tau} I_{\tau \in (t_{m-1}, t_m)}\right] \\ &= \int_{c_1, \dots, c_{m-1}}^{\infty} \left(\prod_{i=1}^{m-1} \left(1 - \exp\left(-\frac{2(c_{i-1} - x_{i-1})(c_i - x_i)}{t_i - t_{i-1}}\right)\right) \right) \\ & \quad \left(\prod_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \right) dx_1 \dots dx_{m-1}, \end{aligned} \quad (\text{A.3})$$

$$e^{-\alpha t_{m-1}} L(t_m - t_{m-1}, \alpha; a_m, c_{m-1} - x_{m-1})$$

where $L(\cdot)$ is the function from equation (A.2) and $\alpha > 0$.

Proof

The proof can be found in Zaeviski (2020), theorem 4.1. $\square$

Proposition A.4. If $z > c(T)$ and the function $c(\cdot)$ is linear, then

$$\begin{aligned} & V(\alpha, z, T; c(\cdot)) \equiv \mathbb{E}\left[e^{\alpha B_T} I_{B_T < z, \Phi_T = 1}\right] = \\ &= \exp\left(\frac{T\alpha^2}{2}\right) \left[\begin{aligned} & N\left(\frac{z - T\alpha}{\sqrt{T}}\right) - N\left(\frac{c(T) - T\alpha}{\sqrt{T}}\right) \\ & - e^{2b(\alpha - a)} \left(N\left(\frac{z - T\alpha - 2b}{\sqrt{T}}\right) - N\left(\frac{c(T) - T\alpha - 2b}{\sqrt{T}}\right) \right) \end{aligned} \right]. \end{aligned} \quad (\text{A.4})$$

Proof

The proof can be found in [Zaevski \(2020\)](#), theorem 3.2. $\square$

Corollary A.1. We have for $z > c(T)$

$$\begin{aligned} U(z, T; c(\cdot)) &\equiv \mathbb{P}(B_T < z, \Phi_T = 1) = N\left(\frac{z}{\sqrt{T}}\right) - N\left(\frac{c(T)}{\sqrt{T}}\right) \\ &\quad - e^{-2ba} \left[N\left(\frac{z - 2b}{\sqrt{T}}\right) - N\left(\frac{c(T) - 2b}{\sqrt{T}}\right) \right]. \end{aligned} \quad (\text{A.5})$$

Proof

The result coincides with Formula (A.4) for $\alpha = 0$. $\square$

Proposition A.5. If the boundary function is piecewise linear, then Formulas (A.4) and (A.5) turn to

$$\begin{aligned} &\mathbb{E}\left[e^{\alpha B_T} I_{B_T < z, \Phi_T = 1}\right] \\ &= \int_{c_1, \dots, c_{n-1}}^{\infty} \left(\prod_{i=1}^{n-1} \left(1 - \exp\left(-\frac{2(c_{i-1} - x_{i-1})(c_i - x_i)}{t_i - t_{i-1}}\right) \right) \right) \\ &\quad \left(\prod_{i=1}^{n-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \right) \\ &\quad \left(e^{\alpha x_{n-1}} V(\alpha, z - x_{n-1}, t_n - t_{n-1}; c_{n-1}(\cdot) - x_{n-1}) \right) dx_1 \dots dx_{n-1}, \end{aligned} \quad (\text{A.6})$$

and

$$\begin{aligned} &\mathbb{P}(B_T < z, \Phi_T = 1) \\ &= \int_{c_1, \dots, c_{n-1}}^{\infty} \left(\prod_{i=1}^{n-1} \left(1 - \exp\left(-\frac{2(c_{i-1} - x_{i-1})(c_i - x_i)}{t_i - t_{i-1}}\right) \right) \right) \\ &\quad \left(\prod_{i=1}^{n-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \right) \\ &\quad \left(e^{\alpha x_{n-1}} U(z - x_{n-1}, t_n - t_{n-1}; c(\cdot) - x_{n-1}) \right) dx_1 \dots dx_{n-1} \end{aligned} \quad (\text{A.7})$$

where the functions $V(\cdot)$ and $U(\cdot)$ are given by equations (A.4) and (A.5).

Proof

The proof of the first statement can be found in [Zaevski \(2020\)](#), theorem 4.2. The second formula can be obtained for $\alpha = 0$. $\square$

The following formulas can be find in [Borodin and Salminen \(2015\)](#), page 223, (2.0.1).

Proposition A.6. Let ζ be the first hitting time of a Brownian motion with drift μ to the level a . Then the Laplace transform is

$$\mathbb{E}[e^{-y\tau}I_{\tau<\infty}] = \begin{cases} e^{-(\sqrt{\mu^2+2y}-\mu)a} & \text{for } a > 0 \\ e^{(\sqrt{\mu^2+2y}+\mu)a} & \text{for } a < 0. \end{cases} \quad (\text{A.8})$$

Appendix 2 Finite difference terms

$$\begin{aligned} F_t &= \frac{F(m-1, n) - F(m, n)}{\Delta t} \\ F &= \frac{F(m-1, n) + F(m, n)}{2} \\ F_x &= \frac{F(m-1, n) - F(m-1, n-1) + F(m, n) - F(m, n-1)}{2\Delta x} \\ F_{xx} &= \frac{F(m-1, n+1) - 2F(m-1, n) + F(m-1, n-1)}{2(\Delta x)^2} \\ &\quad + \frac{F(m, n+1) - 2F(m, n) + F(m, n-1)}{2(\Delta x)^2}. \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} 0 &= \frac{F(m-1, n) - F(m, n)}{\Delta t} + \\ &\quad + \frac{1}{2}rx_n \frac{F(m-1, n) - F(m-1, n-1) + F(m, n) - F(m, n-1)}{\Delta x} \\ &\quad + \frac{1}{4}\sigma^2 x_n^2 \left(\frac{F(m-1, n+1) - 2F(m-1, n) + F(m-1, n-1)}{(\Delta x)^2} \right. \\ &\quad \left. + \frac{F(m, n+1) - 2F(m, n) + F(m, n-1)}{(\Delta x)^2} \right) \\ &\quad - \frac{1}{2}r(F(m-1, n) + F(m, n)). \end{aligned} \quad (\text{B.2})$$

(1) If $n = l_m + 1$, then

$$\begin{aligned}
& F(m, n) \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
& - F(m, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
& = F(m-1, n-1) \left(-\frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
& + F(m-1, n) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
& + F(m-1, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
& - F(m, l_m) \left(\frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right)
\end{aligned} \tag{B.3}$$

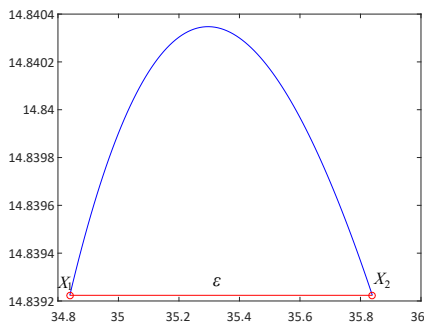
(2) If $l_m + 1 < n < k_m - 1$, then

$$\begin{aligned}
& F(m, n-1) \left(\frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
& + F(m, n) \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
& - F(m, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
& = F(m-1, n-1) \left(-\frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
& + F(m-1, n) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
& + F(m-1, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2}.
\end{aligned} \tag{B.4}$$

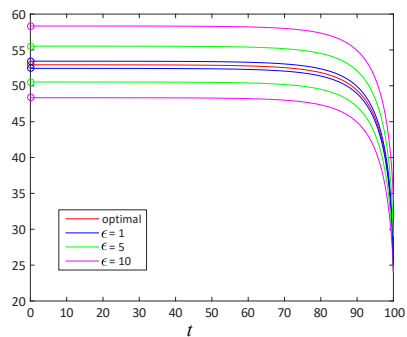
(3) If $n = k_m - 1$, then

$$\begin{aligned}
 & F(m, n-1) \left(\frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
 & + F(m, n) \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
 & = F(m-1, n-1) \left(-\frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
 & + F(m-1, n) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
 & + F(m-1, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
 & + F(m, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2}.
 \end{aligned} \tag{B.5}$$

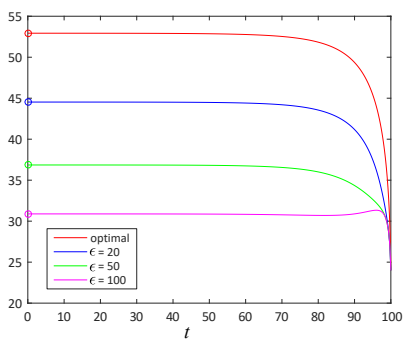
CFRI



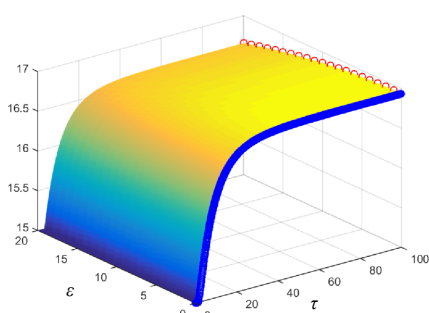
(a) deriving the boundaries



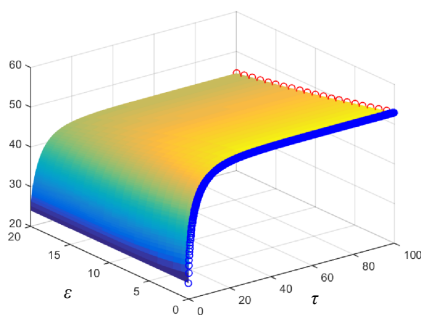
(b) $\epsilon \in \{1, 5, 10\}$



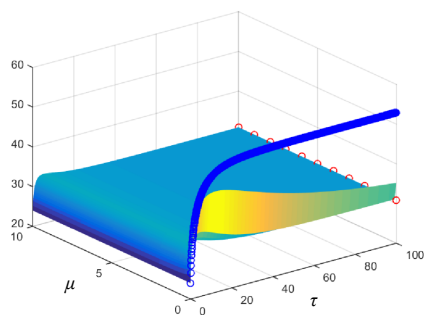
(c) $\epsilon \in \{20, 50, 100\}$



(d) prices, $\epsilon \in (0, 20)$, $t \in (0, 100)$



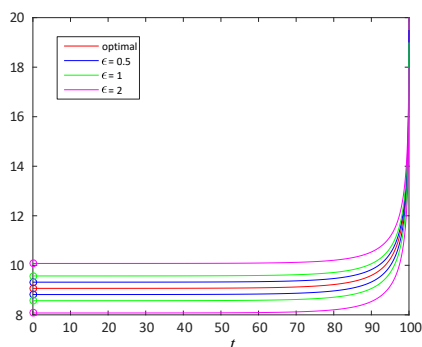
(e) lower boundary surface



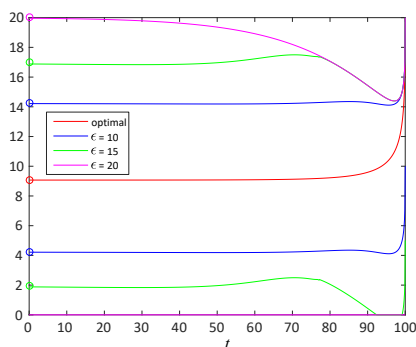
(f) time-dependent ϵ

Source(s): Author's own work

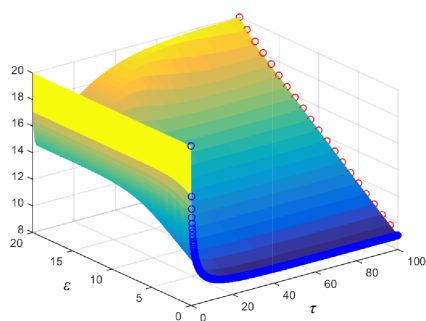
Figure A1. Call ϵ -optimal strip $r = 0.01$, $\lambda = 0.05$, $\sigma = 0.3$, $K = 20$, $T = 50$



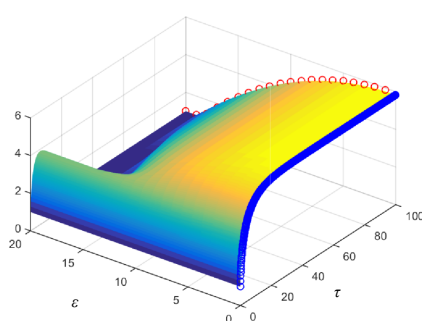
(a) $\epsilon \in \{0.5, 1, 2\}$



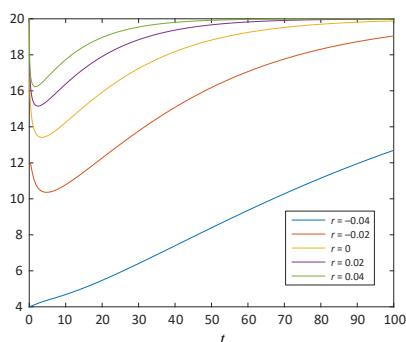
(b) $\epsilon \in \{10, 15, 20\}$



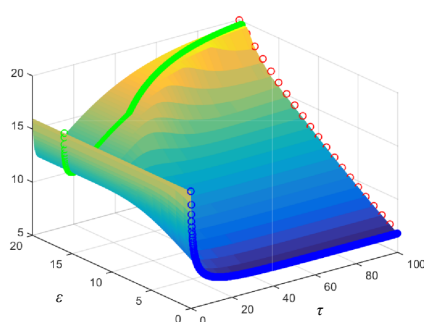
(c) ϵ -boundary surface, $r = 0.01$



(d) prices, $\epsilon \in (0, 2)$, $t \in (0, 3)$



(e) critical values



(f) ϵ -boundary surface, $r = -0.01$

Source(s): Author's own work

Figure A2. Put ϵ -optimal strip $r = 0.01$, $\lambda = 0.05$, $\sigma = 0.3$, $K = 20$

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